HUT, TUT, LUT, OU, ÅAU / Engineering departments Series A

Entrance examination in mathematics May 27, 2003

Instructions. *Reserve a separate page for each problem.* Give your *solutions in a clear form including intermediate steps.* Write a clean copy of the solution if needed. *Cross out discarded solutions*, in case of two solutions, only the weaker one will be credited.

A1. 43 % of the population of a town and its surrounding rural district lived in the town. During the following five years the population of the town increased by 12.4 % and the population of the rural district by 2.6%.

a) By what percentage did the population of the region in question increase?

b) What percentage of the population lived in the town at the end of the period?

A2. Hot water is poured into a thermos bottle, after which the bottle is closed. The temperature y (in Celsius degrees) of the water in the bottle obeys the formula

$$y = 20 + 80e^{-t/1200}$$

where t is the number of minutes after the closing of the bottle.

a) Find the temperature of the water in the bottle after 8 hours. Answer with an accuracy of one degree.

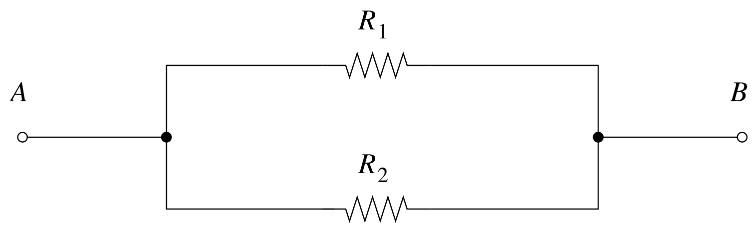
b) How long is the temperature of the water at least 64 °C? Answer in hours and minutes with an accuracy of one minute.

c) Find the cooling velocity $\frac{dy}{dt}$ in the bottle after 8 hours. Give your answer in °C/h.

A3. Find the area of the finite region bounded by the straight line $y = x$, the curve $y = \frac{1}{7x+6}$ and the y-axis.

A4. The circuits in the figures are fully functioning if the current runs from point A to point B along both of the paths. It is partly functioning if the current runs from point A to point B along only one of the paths. We assume that all malfunctioning is caused by the resistances in the circuit. The probability that a resistance of type R_1 is defective is $p_1 = 0.09$. A resistance of type R_2 is defective with the probability $p_2 = 0.17$. We further assume that the functionality or defectiveness of each resistance is independent of the functionality or defectiveness of the other resistances in the circuit.

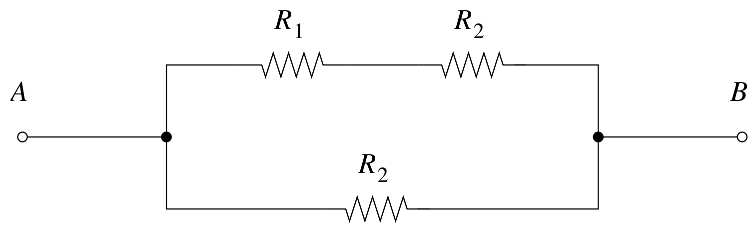
a) Find the probability that circuit 1 is fully functioning.



Circuit 1

b) Find the probability that circuit 2 is not fully functioning.

c) Find the probability that circuit 2 is partly functioning.



Circuit 2

A5. A semicircular marketplace has a radius of 30 m. Its straight side borders a street. A platform with a rectangular bottom is constructed on the marketplace. The edge of the platform closest to the street is parallel to and 10 meters from the street.

Find the maximum of the area of the platform with an accuracy of one square meter.

A6. The length of the hour hand of an old tower clock is 1 meter. The length of the minute hand is r meter.

Find the distance between the tips of the hands, when it is 4 o'clock.

At what speed do the tips of the hands approach each other, when it is 4 o'clock?

MATHEMATICS 2003 **problem 1**

In the solution the populations are given as percentages of a basic value which in a) is the population of the region at the beginning and in b) correspondingly the population of the region at the end of the period.

population at the beginning : 43% + 57% = 100%

population at the end :

$$1.124 \cdot 43\,\% \;+\; 1.026 \cdot 57\,\% \;=$$

$$= 48.332\,\% \;+\; 58.482\,\% \;=$$

$$= 106.814\,\%$$

Thus

a) the population increased by **6.8 %** ,

b) the share of town residents at the end:

$$\frac{48.332}{106.814} = 0.4524\ldots \approx \mathbf{45\,\%}\,.$$

problem 2

8 hours = 480 minutes

a) $y = 20 + 80e^{-480/1200} = 73.62\ldots \approx \mathbf{74}\ {}^{o}\mathrm{C}$

b) $20 + 80e^{-t/1200} \geq 64 \iff -\frac{t}{1200} \geq \ln\left(\frac{44}{80}\right)$

$$\iff t \leq -1200\ln\left(\frac{44}{80}\right) = 717.404\ldots \quad \text{(min)}$$

$$\approx \mathbf{11\ h\ \ 57\ min}$$

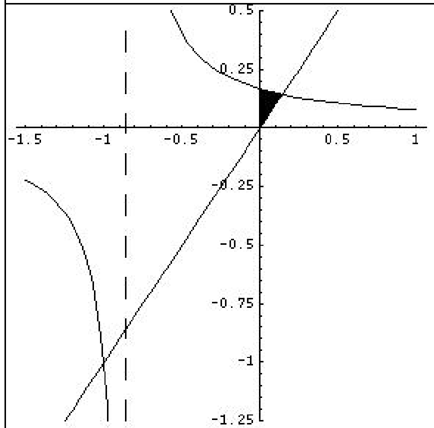
c) $\frac{dy}{dt} = -\frac{80}{1200}\, e^{-\frac{t}{1200}}$

Cooling velocity at t = 480 :

$$\frac{dy}{dt} = -\frac{80}{1200}\, e^{-\frac{480}{1200}} = -0.04468\ldots \ {}^{o}\mathrm{C/min}$$

$$= -2.681\ldots \ {}^{o}\mathrm{C/h} \approx \mathbf{-2.7}\ {}^{o}\mathrm{C/h}$$

MATHEMATICS 2003 **problem 3**



point of intersection: $x = \frac{1}{7x+6} \Leftrightarrow 7x^2 + 6x - 1 = 0$

$\Leftrightarrow \quad x = \frac{1}{7}$ (the root $x = -1$ is irrelevant!)

$$\text{Area} = \int_0^{\frac{1}{7}} \left(\frac{1}{7x+6} - x \right) dx =$$

$$= \frac{1}{7}(\ln(1+6) - \ln 6) - \frac{1}{2} \cdot \frac{1}{7^2} =$$

$$= \mathbf{\frac{1}{7}\ln\left(\frac{7}{6}\right) - \frac{1}{98}} \qquad (= 0.01181\ldots)$$

problem 4

a) P('fully functioning') = $(1 - p_1)(1 - p_2)$ =

= 0.91 · 0.83 = 0.7553 ≈ **0.76**

b) P('not fully functioning') = 1 – P('fully functioning') =

$= 1 - (1 - p_1)(1 - p_2)^2 =$

$= 1 - 0.91 \cdot 0.83^2 = 0.373101$

≈ **0.37**

c) Let q = P('upper route functions') , then

$q = (1 - p_1)(1 - p_2) = 0.7553$.

Further P('lower route functions') = $1 - p_2$.

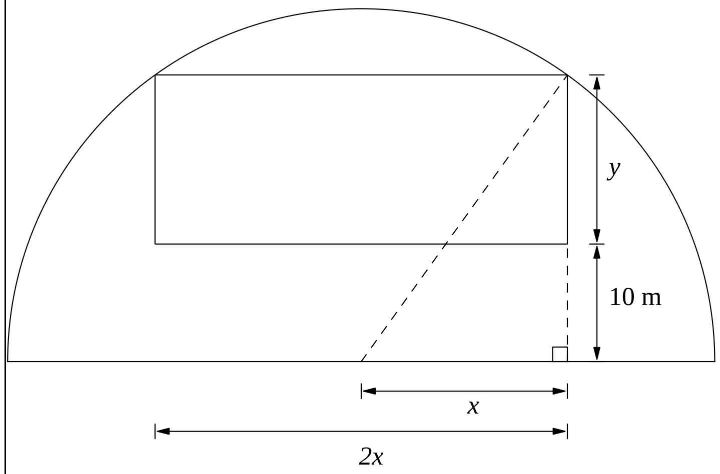
Now P('partly functioning') =

= P('upper functions, lower doesn't') +
+ P('lower functions, upper doesn't') =

$= q\,p_2 + (1 - q)(1 - p_2) = 0.7553 \cdot 0.17 + 0.2447 \cdot 0.83 =$

= 0.331502 ≈ **0.33**

MATHEMATICS 2003 **problem 5**



The area equals $A = 2xy$ (cf. figure), where $x^2 + (y + 10)^2 = 30^2$. Thus $A(y) = 2y\sqrt{30^2 - (y+10)^2}$, $0 \le y \le 20$.
The function A is differentiable and positive except at the end points of the interval , where $A = 0$.

So A attains its maximum value at a zero of A' .

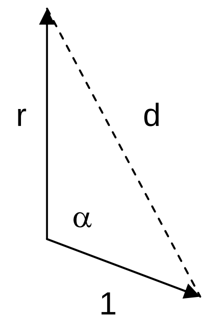
$A'(y) = 2\sqrt{30^2 - (y+10)^2} - \dfrac{2y(y+10)}{\sqrt{30^2 - (y+10)^2}}$, and the condition $A'(y) = 0$ gives the equation $y^2 + 15y - 400 = 0$.

The only solution of this equation on the interval $0 \le y \le 20$ is $y_0 = \frac{5}{2}(-3 + \sqrt{73}) \approx 13.860\ldots$ (m).

Then ($x \approx 18.185\ldots$ and) the area is $A(y_0) = 504.09\ldots \approx$ **504 m²** .

MATHEMATICS 2003 **problem 6**

Let d = distance between the tips of the hands and let α = the angle between the hands.



<u>distance at 4 o'clock</u> : Because $\cos\alpha = \cos\frac{2\pi}{3} = -\frac{1}{2}$,

it follows from the cosine theorem that

$d^2 = 1 + r^2 - 2r\cos\alpha = 1 + r^2 + r$. Thus $d = \sqrt{\mathbf{1+r+r^2}}$.

<u>speed of approach</u> : Let 1 hour be the unit of the time variable t so that t = 0 at 4 o'clock. Then d and α can be treated as functions of t .

The minute hand rotates clockwise with velocity 2π (rad/h) and the hour hand correspondingly with velocity $\frac{1}{12}\cdot 2\pi = \frac{\pi}{6}$. Thus, when $t \approx 0$, the angle between the hands changes with velocity $\alpha'(t) = \left(\frac{1}{12}-1\right)2\pi = \frac{-11\pi}{6}$,

in particular $\alpha'(0) = \frac{-11\pi}{6}$. The distance between the hands changes at t = 0 with velocity d'(0) , which can be calculated by differentiating both sides of the formula $d(t)^2 = 1 + r^2 - 2r\cos(\alpha(t))$ and setting t = 0 :

$$2\,d(t)\,d'(t) = 2r\sin(\alpha(t))\cdot\alpha'(t) \quad\Rightarrow\quad d'(0) = \frac{r\cdot\sqrt{3}/2}{\sqrt{1+r+r^2}}\cdot\frac{-11\pi}{6} = -\,\frac{\mathbf{11\sqrt{3}\pi}}{\mathbf{12}}\cdot\frac{\mathbf{r}}{\sqrt{\mathbf{1+r+r^2}}} \qquad \text{(m/h)}$$