HUT, TUT, LUT, OU, ÅAU / Engineering departments

Entrance examination in mathematics May 25, 2004

Instructions. Reserve *a separate page* for each problem. Give your solutions in a clear form *including intermediate steps*. Write a clean copy of the solution if needed. *Cross out discarded solutions*. In case of several solutions for the same problem, only the weakest one will be credited.

1. A certain article was sold in store A with a 20 % discount off the standard price and later with an additional 10 euro cash discount off the reduced price. In another store B there was a 10 % discount off the same standard price and later an additional 20 euro cash discount off the reduced price. After all the discounts the price of the article was 10 % lower in store A than in store B. What was the standard price of the article?

2. The tangent line to the graph of the function $f(x) = x^2 + 3x$ at the point $(x_0, f(x_0))$ also passes through the point $(2,6)$. Determine all possible values of x_0.

3. A particle moves in the plane so that its coordinates at time t are

$$\begin{cases} x = \cos t + 4\sin t \\ y = -2\cos t + 2\sin t. \end{cases}$$

What is the shortest distance from the origin to the particle?

4. An electric motor is started at time $t = 1$. Because of a defect in the current supply the power of the motor is not constant, but varies with time t according to the formula

$$P(t) = 4000\left(\frac{1}{t} - \frac{1}{t^2}\right).$$

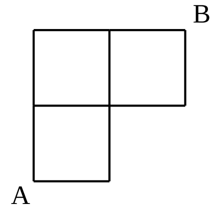
During which time interval $[t_0, t_0+1]$ of 1 second in duration is the motor doing the maximum amount of work?

Additional information: A motor with power $P = P(t)$ does during a time interval $[t_1, t_2]$ the work W which can be calculated as the integral

$$W = \int_{t_1}^{t_2} P(t)\, dt.$$

All quantities are given in the SI-system, so the problem can be treated without units.

5. The adjacent diagram shows a sketch of a part of the streetmap of a city. A person walks daily from crossing A to crossing B along a shortest possible route, so the length of the journey is 4 units. At crossings where two directions give a shortest route the person chooses the direction by tossing a coin.



   a) Draw (separate) diagrams of all possible shortest routes and determine the probability for each of them to be chosen.

   b) Another person walks from crossing B to crossing A and chooses a shortest possible route in the same fashion. Both persons start walking simultaneously and walk with the same speed. With what probability will they meet each other half way?

6. An infinite number of balls with radii $r_1 > r_2 > r_3 > \ldots$ etc. are in a row on a table. Except for the first ball each touches two others and furthermore the midpoints of the balls all lie on the same line (in space).

   a) Determine an expression for the radius r_n as a function of r_{n-1} and r_{n+1} $(n = 2, 3, \ldots)$.

   b) Assume that $r_1 = 3$ and $r_3 = 1$. What is then the sum of the volumes of all the balls?

MATH (engineers) 2004 **problem 1**	**problem 2**

MATH (engineers) 2004 problem 1

Let x be the standard price ; then the discounted prices are

$0.8x - 10$ in store A,
$0.9x - 20$ instore B.

From the assumption we have

$$0.8x - 10 = 0.9(0.9x - 20)$$

$$\Rightarrow \quad 0.01x = 8$$

$$\Rightarrow \quad \mathbf{x = 800} \quad \text{(euros)}$$

problem 2

The equation of the tangent line at $(x_0 , y_0) = (x_0 , f(x_0))$ is

$$y - y_0 = f'(x_0)(x - x_0)$$

or

$$y - (x_0^2 + 3x_0) = (2x_0 + 3)(x - x_0).$$

The point $(x,y) = (2,6)$ satisfies the equation

$\Rightarrow$

$$6 - x_0^2 - 3x_0 = (2x_0 + 3)(2 - x_0)$$

$$= 4x_0 - 2x_0^2 + 6 - 3x_0$$

or

$$x_0^2 - 4x_0 = 0 ,$$

so the solution is

$$\mathbf{x_0 = 0} \quad \text{or} \quad \mathbf{x_0 = 4} .$$

MATH (engineers) 2004

problem 3

At time t the distance of the particle from the origin is

$$r(t) = \sqrt{x^2 + y^2}$$

$$= \sqrt{(\cos t + 4\sin t)^2 + (-2\cos t + 2\sin t)^2}$$

$$= \sqrt{5\cos^2 t + 20\sin^2 t}$$

$$= \sqrt{5 + 15\sin^2 t}$$

($\cos^2 t + \sin^2 t = 1$).

Because the minimum value of $\sin^2 t$ is $\sin^2 0 = 0$ the shortest distance is

$$r(0) = \sqrt{5}$$

(of course, $t = \pm\pi$, $\pm 2\pi$ etc. give the same distance).

problem 4

Let $t_0 \geq 1$. The work done by the motor during the time interval $[t_0 , t_0 + 1]$ is $4000\, w(t_0)$ where

$$w(t_0) = \int_{t_0}^{t_0+1} \left(\frac{1}{t} - \frac{1}{t^2}\right) dt = \Big/_{t_0}^{t_0+1} \ln t + \frac{1}{t}$$

$$= \ln(t_0+1) - \ln t_0 + \frac{1}{t_0+1} - \frac{1}{t_0} .$$

It is sufficient to find the maximum point of the function w :

$$w'(t_0) = \frac{1}{t_0+1} - \frac{1}{t_0} - \frac{1}{(t_0+1)^2} + \frac{1}{t_0^2} = \ldots = \frac{-t_0^2 + t_0 + 1}{(t_0+1)^2 t_0^2} .$$

Thus $w'(t_0) = 0 \Leftrightarrow -t_0^2 + t_0 + 1 = 0 \Leftrightarrow t_0 = \frac{1+\sqrt{5}}{2} \approx 1.618$

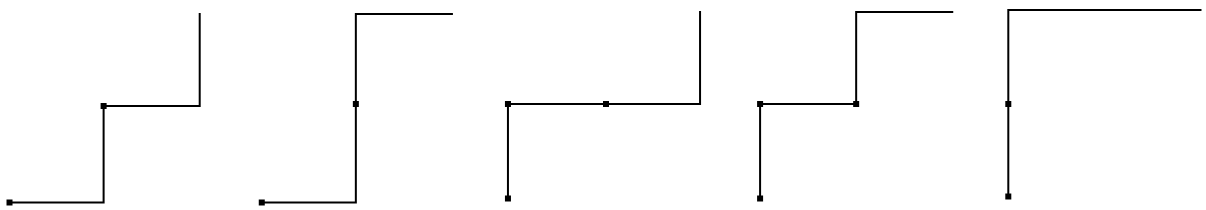
(the second root is negative).

This is really the maximum point of w because $w'(t) > 0$ for $1 \leq t < t_0$ and $w'(t) < 0$ for $t > t_0$.

Answer: $\left[\frac{1+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right] \approx [1.618 , 2.618]$.

MATH (engineers) 2004 **problem 5**

a) There are 5 shortest routes; diagrams and probabilities:



$p_1 = \frac{1}{4}$ $p_2 = \frac{1}{4}$ $p_3 = \frac{1}{8}$ $p_4 = \frac{1}{8}$ $p_5 = \frac{1}{4}$

Motivation: The probability of a certain route equals $\frac{1}{2^n}$ where n is the number of coin tosses on that route (marked in the diagram).

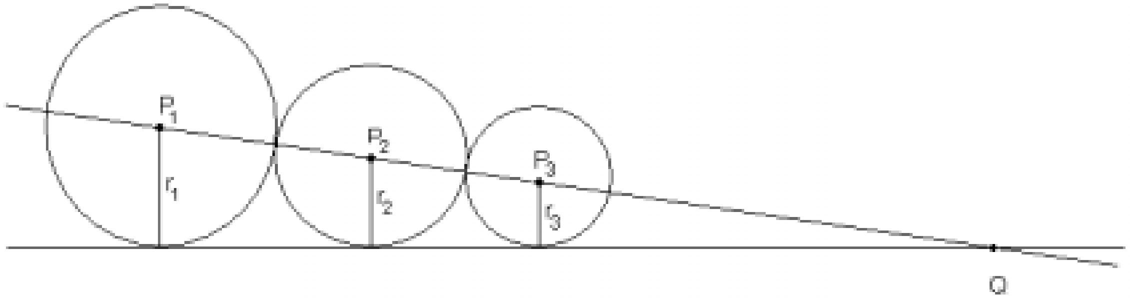
b) The persons meet each other - if they do - either at the upper left corner or at the midpoint. The person who walks from A to B passes the upper corner with probability $p_5 = \frac{1}{4}$ and the midpoint with probability

$p_1 + p_2 + p_3 + p_4 = \frac{3}{4}$.

From the symmetry it follows that for the person walking from B to A the corresponding probabilities are just the same. Thus

$$P(\text{'they meet'}) = P(\text{'both pass the upper corner'}) + P(\text{'both pass the midpoint'}) = \frac{1}{4}\cdot\frac{1}{4}+\frac{3}{4}\cdot\frac{3}{4} = \frac{5}{8}.$$

MATH (engineers) 2004 **problem 6**



Consider three adjacent balls (cf. diagram) with radii r_{n-1} , r_n ja r_{n+1} ; to make the solution more easily readable we denote these by r_1 , r_2 , r_3 . The midpoints P_1 , P_2 , P_3 of the balls are on a straight line; let Q be the point where this line meets the surface of the table. Let c_1 , c_2 , c_3 be the hypothenuses of the three similar right-angled triangles in the diagram (i.e. c_i = distance of P_i from Q , i = 1,2,3).

a) By similarity we have $\frac{c_1}{r_1} = \frac{c_2}{r_2} = \frac{c_3}{r_3}$, whence (express the distances $|P_1P_2|$ and $|P_2P_3|$ in two ways in terms of the radii and c_2) $r_1 + r_2 = c_1 - c_2 = (\frac{r_1}{r_2} - 1)c_2 = (r_1 - r_2)\frac{c_2}{r_2}$ and correspondingly $r_2 + r_3 = \ldots = (r_2 - r_3)\frac{c_2}{r_2}$.

Solving both equations for $\frac{c_2}{r_2}$ gives $\frac{r_1 + r_2}{r_1 - r_2} = \frac{r_2 + r_3}{r_2 - r_3} \Rightarrow \ldots \Rightarrow r_2^2 = r_1 r_3$.

With the original notation: the sought expression is

$$r_n = \sqrt{r_{n-1} r_{n+1}} .$$

part 6b) From the solution of part a) it follows that $\frac{r_{n+1}}{r_n} = \frac{r_n}{r_{n-1}}$ for all n = 2, 3, ... , i.e. there is a constant α such that the ratio of the radii of two adjacent balls is

$$\alpha = \frac{r_n}{r_{n-1}}$$

for all n .

If $r_1 = 3$ and $r_3 = 1$ then $r_2 = \sqrt{3 \cdot 1} = \sqrt{3}$, so that $\alpha = \frac{1}{\sqrt{3}}$.

Thus the radii of the balls are r_1 , $r_2 = \alpha r_1$, $r_3 = \alpha^2 r_1$, ... , and the sum of the volumes is

$$\frac{4\pi}{3}(r_1^3 + r_2^3 + r_3^3 + \dots) = \frac{4\pi}{3} r_1^3 (1 + \alpha^3 + (\alpha^3)^2 + \dots) =$$

$$= \frac{4\pi}{3} \cdot 3^3 \cdot \frac{1}{1-\alpha^3} \qquad \text{(the sum of a geometric series)}$$

$$= \frac{4\pi \cdot 27}{3} \cdot \frac{1}{1-\left(\frac{1}{\sqrt{3}}\right)^3} = \frac{108\sqrt{3}\pi}{3\sqrt{3}-1}$$