Entrance examination in mathematics May 31, 2005

Instructions. Reserve a *separate page* for each problem. Give your solutions in a clear form *including intermediate steps*. Write a clean copy of the solution if needed. *Cross out discarded solutions* (in case of several solutions of the same problem, only the weakest one will be credited).

A1. At the beginning of the year 2004, about 64.9 % of the population of the Tampere subregion were living in the city of Tampere, and 35.1 % in the six neighbouring municipalities. Suppose that the population of the city of Tampere continues rising by 0.9 % a year and that of the neighbouring municipalities by 1.7 % a year.
a) How large part of the population of the Tampere subregion will be living in the city at the beginning of the year 2024? Give the answer in percentages with one decimal place.
b) At the beginning of which year will the population of the city of Tampere be smaller than that of the neighbouring municipalities?

A2. a) Determine the intersection points of the plane curves $y = 25x^3$ and $x = 5y^2$.
b) Determine the area of the region bounded by the two curves in item a).

A3. A regular cross is drawn inside a circle of radius R, as in Figure 1. Which value of the angle α maximizes the area of the cross? Give the answer in degrees with one decimal place.

A4. In the process of induction hardening two sides of a rectangular metal sheet are heated by the electric current generated by an oscillating magnetic field. We examine the temperature at eleven points on the sheet (cf. Figure 2). At eight points on the boundary the temperature has been measured to be either 20 °C or 710 °C, as in the figure. Calculate approximations for the temperatures at the inner points P_1, P_2 and P_3 of the sheet, so that each of these points has the average temperature of the four nearest points on the boundary or inside the sheet. Give the answers in whole degrees.

A5. A simple traffic model is based on the assumptions that pointlike cars drive in a queue with constant speed v and with a constant mutual distance a. Furthermore, the speed v, distance a and speed limit v_0 are thought to be connected by the equation

$$v/v_0 = \min\{1, \sqrt{a/L}\},$$

where L is a constant depending on the driving conditions and traffic culture. On the exit roads the speed limit is $v_0 =$100 km/h. During the rush hour on Friday at 4 p.m. the number of cars heading to the countryside is measured to be 1180 cars/h and at 6 p.m. the corresponding number is 1920 cars/h. What is the speed of the cars driving in the queue a) at 4 p.m., b) at 6 p.m., when the value $L = 80$ m is used?
(Note: $\min\{x,y\}$ denotes the smaller of the numbers x and y.)

A6. Ice hockey teams A and B decide the league championship by playing a series of matches. The first to win four matches will be the champion. If necessary, overtime periods are played so that no match can end in a draw. Suppose that team A wins each match with probability 1/3 and that the results of different matches are independent of each other.
a) What is the probability that at least five matches are needed before the champion is found?
b) What is the probability that team A wins the championship?

Appendix: Table of Formulas

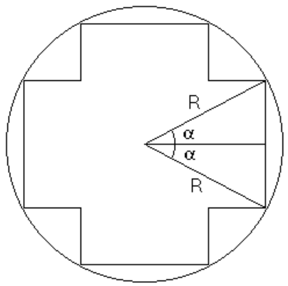


Figure 1

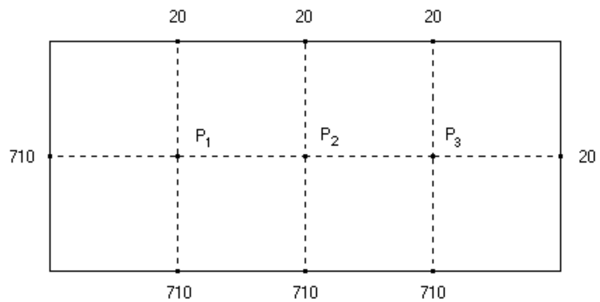


Figure 2

INSMAT 2005 **assignment 1**

Let A > 0 be the number of inhabitants at the beginning of year 2004.

a) At the beginning of year 2024 the number of inhabitants

of the city of Tampere = x = $1.009^{20} \cdot 64.9$ A = 77.63... A,
of the neighbouring communities = y = $1.017^{20} \cdot 35.1$ A = 49.17... A.

$\frac{x}{x+y}$ = 0.6122... $\Rightarrow$ **61,2 %** of the population is living in the city.

b) After n years the city of Tampere has less inhabitants if

$$1.009^{n} \cdot 64.9 \text{ A} < 1.017^{n} \cdot 35.1 \text{ A} \quad \Leftrightarrow \quad \left(\frac{1.017}{1.009}\right)^{n} > \frac{64.9}{35.1}$$

$$\Leftrightarrow \quad n > \frac{\ln(64.9/35.1)}{\ln(1.017/1.009)} = 77.82\ldots \quad \Rightarrow \quad n = 78 \text{ (the smallest integer)}$$

So the city of Tampere has for the first time less inhabitants after 78 years, i.e. at the beginning of year **2082** .

INSMAT 2005 **assignment 2**

a) Solve the system of equations $\begin{cases} y = 25x^3, \\ x = 5y^2 \end{cases}$:

$y = 25\cdot(5y^2)^3 = 5^5y^6 \iff y(1-5^5y^5) = 0 \iff y = 0$ or $y = 1/5$;

with $x = 0$ or $x = 1/5$ respectively.

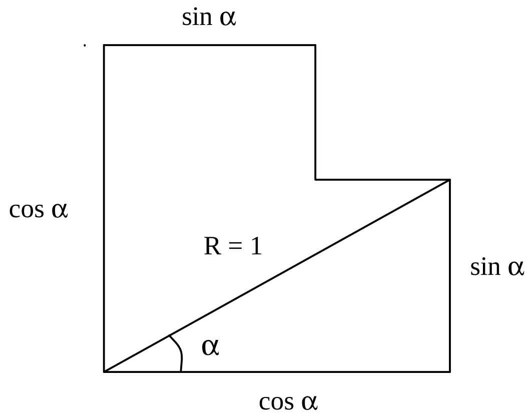
Thus the intersection points are **(0,0)** and $(\mathbf{\frac{1}{5}}, \mathbf{\frac{1}{5}})$.

b) The region in question is A : $25x^3 \le y \le \sqrt{\frac{x}{5}}$, $0 \le x \le \frac{1}{5}$, with area

$$a(A) = \int_0^{\frac{1}{5}} \left(\sqrt{\frac{x}{5}} - 25x^3 \right) dx = \Big/_0^{\frac{1}{5}} \left(\frac{2x^{3/2}}{3\sqrt{5}} - \frac{25x^4}{4} \right) = \mathbf{\frac{1}{60}}$$

INSMAT 2005 **assignment 3**

Obviously $0 \le \alpha \le \pi/4$, moreover it is not a restriction to suppose that $R = 1$. A quarter of the cross (cf. figure) is a square with side $\cos\alpha$, from the corner of which a smaller square with side $\cos\alpha - \sin\alpha$ has been removed.



The area of the cross, as a function of α, attains its maximum at the maximum point of the function

$$f(\alpha) = \cos^2\alpha - (\cos\alpha - \sin\alpha)^2 =$$
$$= 2\sin\alpha\cos\alpha - \sin^2\alpha .$$

Now $f'(\alpha) = 2\cos^2\alpha - 2\sin^2\alpha - 2\sin\alpha\cos\alpha = 2\cos 2\alpha - \sin 2\alpha$,

so $f'(\alpha) = 0 \iff \dfrac{\sin 2\alpha}{\cos 2\alpha} = 2$ or $\tan 2\alpha = 2$.

Thus $2\alpha =$ arctan 2 = 1.107... (radians) = $63.434...^{\circ}$, i.e. $\alpha = 31.717...^{\circ}$.

This is the maximum point (on the interval $0 \le \alpha \le 45^{\circ}$) which can be seen for example by observing that the derivative changes sign from + to – at this point.

Answer: $\alpha =$ **31.7**$^{\circ}$.

INSMAT 2005 **assignment 4**

The temperatures x_1, x_2, x_3 at the points P_1, P_2, P_3 satify the linear system

$$\begin{cases} x_1 = \frac{1}{4}(20+2\cdot 710+x_2) \\ x_2 = \frac{1}{4}(20+x_1+710+x_3) \\ x_3 = \frac{1}{4}(2\cdot 20+x_2+710) \end{cases}$$

$$\begin{cases} 4x_1 - x_2 = 1440 \\ -x_1 + 4x_2 - x_3 = 730 \\ - x_2 + 4x_3 = 750 \end{cases} .$$

The solution of the system is

$$\begin{cases} x_1 = 1805/4 \approx \mathbf{451} \\ x_2 = \mathbf{365} \\ x_3 = 1115/4 \approx \mathbf{279} \end{cases}$$

INSMAT 2005 **assignment 5**

Let v = the velocity of the queue and a = the distance between two consecutive cars.
If N is the number of cars per hour then Na = v or a = v/N.
From the equation $v/v_0 = \sqrt{a/L}$ we get

$$\frac{v^2}{v_0^2} = \frac{a}{L} = \frac{v}{NL} \quad \text{or} \quad v = \frac{v_0^2}{NL} \qquad (1)$$

(v = 0 is inadequate) where v_0 = 100 km/h and L = 80 m = 0.08 km (and the dimension of N is 1/h).

a) (at 16 o'clock) Now N = 1180 and formula (1) gives $v = \frac{100^2}{1180 \cdot 0.08}\frac{km}{h}$ = 105.93... km/h,

which is impossible because of the restriction $v/v_0 \le 1$ (which means that the speed limit is respected).

Thus $v/v_0 = 1$ or

v = **100 km/h**.

b) (at 18 o'clock) Now N = 1920 and from formula (1) we get

v = **65.1 km/h**.

INSMAT 2005 **assignment 6**

a) P("4 games suffice") = P("A wins 4 consecutive games") + P("B wins 4 consecutive games") =

$$\left(\frac{1}{3}\right)^4 + \left(\frac{2}{3}\right)^4 = \frac{17}{81} \quad \Rightarrow \quad \text{P("at least 5 games are needed")} = 1 - \frac{17}{81} = \mathbf{\frac{64}{81}} .$$

b) If A wins the championship then A must win 4 games while B wins n games where $0 \le n \le 3$.
Necessarily A wins the last game. In the first 3+n games the order of victories is arbitrary; the number of different orderings is $\binom{3+n}{n}$ (binomial coefficient).

Thus P("A wins 3, B wins n") = $\binom{3+n}{n}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^n$, so for different values of n we get (observing the probability 1/3 that A wins the last game)

P("A wins 4 of 4") = $\left(\frac{1}{3}\right)^4 = \frac{1}{81}$, P("A wins 4 of 5") = $\binom{4}{1}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right) = \frac{8}{243}$

P("A wins 4 of 6") = $\binom{5}{2}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 = \frac{40}{729}$, P("A wins 4 of 7") = $\binom{6}{3}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^3 = \frac{160}{2187}$.

Adding these we get P("A wins the championship") = $\mathbf{\frac{379}{2187}}$ = 0.173...